Curvature of attached shock waves in steady axially symmetric flow

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An electronic computer has been employed to calculate the ratio between the initial radii of curvature of the attached shock wave and the body for an axially symmetrical body in a uniform supersonic stream. The results are obtained with 4 exact digits for more than 200 cases. They extend results obtained previously (Cabannes 1951) by means of numerical integration.

1. Introduction

We consider a body of revolution placed in a compressible fluid. The fluid possesses at infinity a uniform supersonic velocity $\bar{q}$ parallel to the axis of revolution Ox. A shock wave is formed in front of the body, and limits the region in

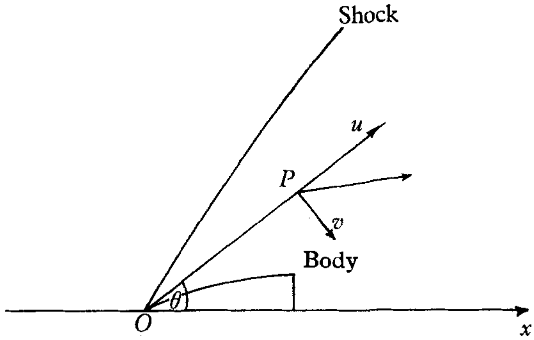

FIGURE 1. Diagram of the flow field.

which the flow is uniform. Viscosity and thermal conductivity are neglected outside the shock. We suppose that the surface of the body is tangential at the axis of revolution to a cone with semi-angle θ_s, and that the angle θ_s and the Mach number M of the upstream flow have been chosen in such a way that the shock wave is attached at the vertex O of the obstacle. We locate the position of a point P in a meridian plane by the polar co-ordinates $OP = r$ and $\angle POx = \theta$ (see figure 1). By means of these co-ordinates, the equation of the obstacle in the neighbourhood of the point O can be written in the form (1) and the equation of the shock wave, in the neighbourhood of the same point, in the form (2), namely,

$$\text{body: } \theta = \theta_s + \frac{r}{2\mathscr{R}} + \dots, \tag{1}$$

$$\text{shock: } \theta = \theta_w + \frac{r}{2R} + \dots. \tag{2}$$

The angle θ_w is determined by the theory of axially symmetric flow (Kopal 1947); it depends on the Mach number M and the angle θ_s. The object of the present paper is to give tables for the determination of the value of the ratio $(R/\mathscr{R})$ of the radii of curvature, at the axis of revolution, of the shock wave and the body; this ratio likewise depends on the Mach number M and the angle θ_s.

2. Equations of motion

We designate by u and v the components of the fluid velocity at a point P in the directions θ and $(\theta+\frac{1}{2}\pi)$, by p and ρ the pressure and density at this point, and by γ the ratio of the specific heats of the fluid. The four functions u, v, p and ρ of the variables r and θ satisfy the following partial differential equations which express the fundamental law of dynamics, the conservation of mass and the conservation of energy:

$$\left.\begin{aligned} u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+\frac{1}{\rho}\frac{\partial p}{\partial r} &= 0,\\ u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}+\frac{1}{\rho r}\frac{\partial p}{\partial \theta} &= 0,\\ \frac{\partial}{\partial r}(r^2\rho u\sin\theta)+\frac{\partial}{\partial \theta}(r\rho v\sin\theta) &= 0,\\ u\frac{\partial}{\partial r}(p\rho^{-\gamma})+\frac{v}{r}\frac{\partial}{\partial \theta}(p\rho^{-\gamma}) &= 0.\end{aligned}\right\} \tag{3}$$

We attempt to satisfy the preceding equations by means of functions expanded in series of whole and increasing powers of r, the coefficients depending only on the variable θ:

$$\left.\begin{aligned} u(r,\theta) &= u_0(\theta)+\frac{r}{R}u_1(\theta)+\ldots,\\ v(r,\theta) &= v_0(\theta)+\frac{r}{R}v_1(\theta)+\ldots,\\ p(r,\theta) &= p_0(\theta)+\frac{r}{R}p_1(\theta)+\ldots,\\ \rho(r,\theta) &= \rho_0(\theta)+\frac{r}{R}\rho_1(\theta)+\ldots.\end{aligned}\right\} \tag{4}$$

By substitution of these expansions into equations (3) and by identification according to successive powers of r, one obtains an infinite set of differential equations. Equations (3) have a first integral, Bernoulli's equation. As the limiting speed q_m is constant in front of the shock and continuous across the shock wave, we have, valid in all the fluid,

$$\frac{2\gamma}{\gamma-1}\frac{p}{\rho}+u^2+v^2 = q_m^2. \tag{5}$$

We also introduce the function $a_0(\theta)$ defined by:

$$a_0^2 = \gamma\frac{p_0}{\rho_0} = \frac{\gamma-1}{2}(q_m^2-u_0^2-v_0^2). \tag{6}$$

The differential equations deduced from equations (3) can be written in the following form. Using given initial conditions, the functions with suffix 0 can be calculated from equations (7), while the functions with suffix 1 can be calculated from equations (8):

$$\left.\begin{aligned} u_0' - v_0 &= 0, \\ v_0'\left(1-\frac{a_0^2}{v_0^2}\right) + u_0\left(1-2\frac{a_0^2}{v_0^2}\right) - \frac{a_0^2}{v_0}\cot\theta &= 0, \\ \frac{\rho_0'}{\rho_0}\left(1-\frac{a_0^2}{v_0^2}\right) + \frac{u_0}{v_0} + \cot\theta &= 0, \\ \frac{p_0'}{p_0} - \gamma\frac{\rho_0'}{\rho_0} &= 0. \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} u_1' v_0 + u_0 u_1 - v_0 v_1 + \frac{a_0^2}{\gamma}\frac{p_1}{p_0} &= 0, \\ \left(\frac{\rho_1}{\rho_0}\right)' + \frac{v_0 v_0'}{a_0^2} + \frac{u_1 v_0 + v_1(2u_0 + v_0')}{a_0^2} & \\ - \frac{\rho_1}{\rho_0}\left(\frac{\rho_1'}{\rho_0} - \frac{u_0}{v_0}\right) + \frac{p_1}{p_0}\left(\frac{\rho_0'}{\rho_0} - \frac{1}{\gamma}\frac{u_0}{v_0}\right) &= 0, \\ \left(\frac{\rho_1}{\rho_0}\right)' + \frac{v_1'}{v_0} + 3\frac{u_1}{v_0} + \frac{v_1}{v_0}\left(\frac{\rho_0'}{\rho_0} + \cot\theta\right) + \frac{\rho_1}{\rho_0}\frac{u_0}{v_0} &= 0, \\ \frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} + (\gamma-1)\frac{u_0 u_1 + v_0 v_1}{a_0^2} &= 0. \end{aligned}\right\} \tag{8}$$

3. Boundary conditions on the body

The body is formed by the stream surface extended from the point 0. By expressing the condition that the differential equation of the stream function,

$$\frac{dr}{u} = \frac{r\,d\theta}{v}, \tag{9}$$

is satisfied by the function (1), one obtains the conditions

$$v_0(\theta_s) = 0, \tag{10a}$$

$$\frac{v_0'(\theta_s)}{2\mathscr{R}} + \frac{v_1(\theta_s)}{R} = \frac{u_0(\theta_s)}{2\mathscr{R}}. \tag{10b}$$

According to the second of equations (7), one has that $v_0'(\theta_s) = -2u_0(\theta_s)$; therefore the condition (10b) can be written in the form

$$\frac{R}{\mathscr{R}} = \frac{2\,v_1(\theta_s)}{3\,u_0(\theta_s)}. \tag{11}$$

4. Conditions on the shock wave

At the shock wave, a certain number of conditions must be satisfied. These conditions, which express the fundamental law of dynamics, the conservation of mass and the conservation of energy, are expressed by equations (12), in which $\bar{c}$, $\bar{p}$ and $\bar{\rho}$ designate the speed of sound, pressure and density in front of the shock while β is the angle which the tangent to the shock wave makes with the axis of revolution. $\mathscr{M}$ designates the Mach number along the normal ($\mathscr{M} = M\sin\beta$).

$$\left.\begin{aligned} u &= \bar{q}\cos\theta + \frac{2\bar{c}}{\gamma+1}\left(\mathscr{M} - \frac{1}{\mathscr{M}}\right)\sin(\beta-\theta), \\ v &= -\bar{q}\sin\theta - \frac{2\bar{c}}{\gamma+1}\left(\mathscr{M} - \frac{1}{\mathscr{M}}\right)\cos(\beta-\theta), \\ \frac{p}{\bar{p}} &= \frac{2\gamma}{\gamma+1}\mathscr{M}^2 - \frac{\gamma-1}{\gamma+1}, \\ \frac{\bar{\rho}}{\rho} &= \frac{2}{\gamma+1}\frac{1}{\mathscr{M}^2} + \frac{\gamma-1}{\gamma+1}. \end{aligned}\right\} \tag{12}$$

The Mach number M is expressed as a function of the speed $\bar{q}$ by

$$M^2 = \frac{2}{\gamma-1}\frac{\bar{q}^2}{q_m^2 - \bar{q}^2}. \tag{13}$$

By expressing the fact that the equations (12) are satisfied identically on the shock wave, one obtains the following values for the functions with suffix 0 and 1 for $\theta = \theta_w$:

$$\left.\begin{aligned} u_0(\theta_w) &= \bar{q}\cos\theta_w, \\ v_0(\theta_w) &= \frac{\gamma-1}{\gamma+1}\frac{\bar{q}^2\cos^2\theta_w - q_m^2}{\bar{q}\sin\theta_w}, \\ \frac{p_0(\theta_w)}{\bar{p}} &= \frac{2\gamma}{\gamma+1}M^2\sin^2\theta_w - \frac{\gamma-1}{\gamma+1}, \\ \frac{\bar{\rho}}{\rho_0(\theta_w)} &= \frac{2}{\gamma+1}\frac{1}{M^2\sin^2\theta_w} + \frac{\gamma-1}{\gamma+1}; \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} u_1 + u_0\tan\theta_w + v_0 &= 0, \\ 2v_1 + \frac{\gamma-7}{\gamma+1}u_0 + \frac{\gamma+3}{\gamma+1}v_0\cot\theta_w &= 0, \\ \frac{p_1}{p_0} &= \frac{\gamma}{\gamma+1}\cot\theta_w - \frac{4\gamma}{\gamma+1}\frac{u_0 v_0}{a_0^2}, \\ \frac{\rho_1}{\rho_0} &= \frac{2\gamma+3}{\gamma+1}\cot\theta_w + 2\frac{\gamma-1}{\gamma+1}\frac{u_0}{v_0}. \end{aligned}\right\} \tag{15}$$

5. Numerical integration

The numerical integration of equations (7) and (8) has been performed with the help of electronic computer gamma of the Faculty of Sciences of Grenoble. The great capacity of the machine and its high velocity of execution have allowed the computation of 209 cases to be performed, corresponding to 15 different bodies. The method of integration adopted is the Runge–Kutta method of fourth order, with intervals equal to one-twentieth of a degree; it seems that the value of the ratio of the curvatures can then be predicted with 4 exact digits. The results, which are given in the following tables,* have been computed with the adiabatic

* For $u_0(\theta_s)/q_m = (1/6)^{\frac{1}{2}} = 0{\cdot}4082$, the speed on the body, at the vertex, is sonic.

index having the value $\gamma = 1\cdot4$. The ratio of the curvatures is negative for the limiting velocity for which the shock wave is detached from the body; it is zero for a given value of the Mach number, which has been computed.

In the case where the angle θ_s is small, it can be verified that the asymptotic formula, given by Rao (1956),

$$\frac{R}{\mathscr{R}} \sim \frac{40}{81}\frac{1}{(\gamma+1)^4}\frac{(M^2-1)^3}{M^{13}}\theta_s^{-7}, \tag{16}$$

is satisfactory for finite values of the Mach number. For higher values of θ_s, the results are exhibited graphically in figure 2.

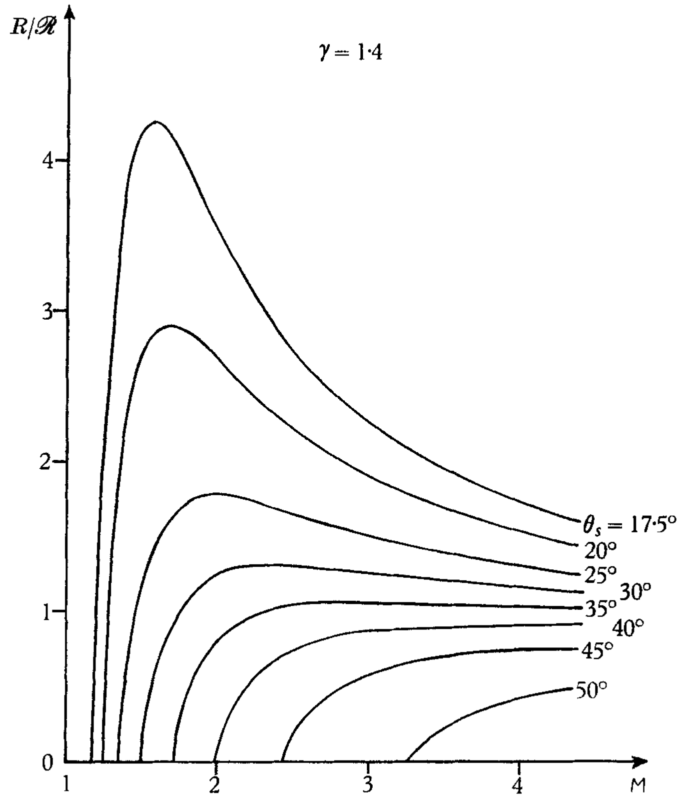

FIGURE 2. Values of $R/\mathscr{R}$ *vs* M for various θ_s.

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$u_0(\theta_s)/q_m$	M	θ_w*	$R/\mathscr{R}$
		$\theta_s = 5°$	
0·35	1·1732	89·224	−0·3234
0·39	1·0215	86·921	−1·4832
0·3913	1·0180	86·441	−1·7700
0·395	1·0128	84·104	−2·2779
0·399	1·0168	80·841	6·0933
0·4	1·0187	80·072	10·4070
0·4082	1·0414	74·131	167·7417
0·55	1·5151	41·363	4759·00
0·6	1·7258	35·482	2196·71
0·65	1·9699	30·597	1172·21
0·7	2·2611	26·372	576·76
0·75	2·6224	22·592	255·98
0·8	3·0961	19·107	103·76
0·85	3·7723	15·792	29·586
0·9	4·8895	12·527	12·9795
0·95	7·4586	9·136	4·3398
0·99	22·6254	5·992	1·3343
		$\theta_s = 7·5°$	
0·36	1·0980	87·324	−0·6790
0·39	1·0334	78·125	0·6794
0·395	1·0420	75·455	6·1709
0·4	1·0529	73·030	15·591
0·4082	1·0766	69·063	43·683
0·45	1·1971	57·012	212·00
0·5	1·3657	47·360	281·44
0·55	1·5555	40·313	223·08
0·6	1·7715	34·729	145·74
0·65	2·0229	30·082	84·598
0·7	2·3244	26·080	46·690
0·75	2·7007	22·537	25·197
0·8	3·1985	19·326	13·516
0·85	3·9177	16·348	7·3241
0·9	5·1329	13·505	3·9809
0·95	8·1036	10·665	2·1041
0·98	16·8180	8·850	1·2883
		$\theta_s = 10°$	
0·3765	1·0535	77·311	−0·8455
0·3810	1·0576	75·347	0·0000
0·4	1·0948	67·957	9·0936
0·4082	1·1190	64·899	15·9125
0·45	1·2398	54·713	42·7685
0·5	1·4109	45·976	49·7619
0·55	1·6049	39·445	39·7939
0·6	1·8271	34·238	27·6345
0·65	2·0872	29·915	18·1499
0·7	2·4009	26·220	11·8199
0·75	2·7956	22·985	7·7432
0·8	3·3231	20·093	5·1675
0·85	4·0951	17·499	3·4898
0·9	5·4526	14·966	2·3434
0·95	9·1471	12·541	1·5017
0·98	19·2181	11·048	1·0663
		$\theta_s = 12·5°$	
0·3	1·3168	86·723	−0·8426
0·3738	1·0902	72·409	0·0249
0·4	1·1429	64·103	5·1224

$u_0(\theta_s)/q_m$	M	θ_w	$R/\mathscr{R}$
		$\theta_s = 12·5°$ (*cont.*)	
0·4082	1·1674	61·593	7·3049
0·45	1·2892	52·804	14·2637
0·5	1·4633	44·923	15·6587
0·55	1·6623	38·934	13·1465
0·6	1·8916	34·144	10·0314
0·65	2·1618	30·176	7·4305
0·7	2·4900	26·802	5·4927
0·75	2·9070	23·869	4·0938
0·8	3·4725	21·265	3·0785
0·85	4·3239	18·903	2·3215
0·9	5·8910	16·709	1·7300
0·95	11·1397	14·606	1·2335
		$\theta_s = 15°$	
0·3	1·2830	84·715	−1·0269
0·35	1·1196	75·231	−0·8898
0·3670	1·1289	70·070	0·0000
0·385	1·1608	64·965	1·6265
0·4	1·1964	61·214	3·1447
0·45	1·3448	51·391	6·6419
0·5	1·5224	44·289	7·2050
0·55	1·7271	38·827	6·3878
0·6	1·9648	34·444	5·2728
0·65	2·2468	30·816	4·2547
0·7	2·5928	27·738	3·4213
0·75	3·0381	25·071	2·7579
0·8	3·6541	22·712	2·2271
0·85	4·6151	20·587	1·7922
0·9	6·5337	18·630	1·4227
0·95	16·8844	16·787	1·0793
		$\theta_s = 17·5°$	
0·3	1·2665	82·113	−1·1618
0·3582	1·1707	68·912	−0·1426
0·3611	1·1743	68·153	0·0000
0·4	1·2551	59·124	2·1327
0·45	1·4062	50·476	3·8682
0·5	1·5881	44·067	4·2223
0·55	1·7996	39·089	3·9217
0·6	2·0472	35·079	3·4297
0·65	2·3438	31·756	2·9338
0·7	2·7122	28·939	2·4909
0·75	3·1947	26·502	2·1100
0·8	3·8809	24·355	1·7717
0·85	5·0080	22·430	1·4945
0·9	7·5793	20·699	1·2427
		$\theta_s = 20°$	
0·3	1·2728	79·220	−1·1975
0·356	1·2271	66·747	−0·0045
0·3561	1·2259	66·670	0·0000
0·4	1·3191	57·697	1·5749
0·4082	1·3452	56·068	1·8450
0·45	1·4737	50·008	2·6155
0·5	1·6609	44·205	2·8858
0·55	1·8806	39·658	2·7762
0·6	2·1404	35·980	2·5168
0·65	2·4555	32·928	2·2497

TABLE 1

* The angles θ_w are given in degrees.

$u_0(\theta_s)/q_m$	M	θ_w	$R/\mathscr{R}$
	θ_s = 20° (*cont.*)		
0·7	2·8531	30·339	1·9759
0·75	3·3862	28·105	1·7346
0·8	4·1742	26·141	1·5129
0·85	5·5710	24·391	1·3591
0·9	9·6230	22·807	1·1805
	θ_s = 22·5°		
0·3	1·3014	76·482	−1·1329
0·3518	1·2840	65·588	0·0000
0·3520	1·2856	65·596	0·0057
0·4	1·3888	56·913	1·2413
0·4082	1·4157	55·363	1·4234
0·45	1·5479	49·920	1·9532
0·5	1·7418	44·643	2·1769
0·55	1·9718	40·476	2·1487
0·6	2·2471	37·092	2·0138
0·65	2·5860	34·279	1·8415
0·7	3·0229	31·894	1·6881
0·75	3·6274	29·837	1·4817
0·8	4·5707	28·036	1·3309
0·85	16·7401	25·002	1·0461
	θ_s = 25°		
0·3	1·3484	74·181	−1·0138
0·3478	1·3487	64·913	−0·0089
0·3481	1·3493	64·841	0·0000
0·3489	1·3504	64·704	0·0174
0·35	1·3534	64·550	0·0427
0·4	1·4653	56·367	1·0250
0·4082	1·4931	55·066	1·1591
0·45	1·6299	50·146	1·5576
0·5	1·8325	45·328	1·7465
0·55	2·0757	41·497	1·7606
0·6	2·3712	38·374	1·6849
0·65	2·7422	35·775	1·5724
0·7	3·2335	33·571	1·4482
0·75	3·9442	31·698	1·3211
0·8	5·1439	30·018	1·2101
0·85	8·1165	28·5546	1·0877
0·9	27·9654	27·626	0·8890
	θ_s = 30°		
0·3	1·4857	71·094	−0·7675
0·3427	1·5058	64·163	0·0000
0·35	1·5180	63·068	0·1194
0·4	1·6446	56·477	0·7645
0·4082	1·6750	55·409	0·8509
0·45	1·8259	51·326	1·1179
0·5	2·0546	47·265	1·2721
0·55	2·3390	44·050	1·3128
0·6	2·6971	41·325	1·2929
0·65	3·1745	39·093	1·2410
0·7	3·8645	37·202	1·1786
0·75	5·0277	35·579	1·1085
0·8	7·8707	34·171	1·0384
0·835	24·0388	32·291	0·8803
	θ_s = 35°		
0·3	1·6797	69·681	−0·5928
0·3391	1·7099	64·361	−0·0047

$u_0(\theta_s)/q_m$	M	θ_w	$R/\mathscr{R}$
	θ_s = 35° (*cont.*)		
0·3395	1·7105	64·316	0·0000
0·35	1·7309	63·009	0·1325
0·4	1·8766	57·548	0·6104
0·4082	1·9115	56·656	0·6745
0·45	2·0875	53·216	0·8793
0·5	2·3633	49·758	1·0112
0·55	2·7228	46·957	1·0635
0·6	3·2112	44·653	1·0673
0·65	3·9340	42·729	1·0473
0·7	5·2059	41·122	1·0160
0·75	8·6893	39·706	0·9780
0·78	24·7546	38·964	0·8756
	θ_s = 40°		
0·3	1·9533	69·453	−0·4848
0·3374	1·9938	65·177	−0·0107
0·3381	1·9982	65·147	0·0000
0·34	1·9993	64·902	0·0133
0·35	2·0235	63·863	0·1217
0·4	2·2027	59·282	0·5052
0·4082	2·2463	58·527	0·5565
0·45	2·4712	55·606	0·7275
0·5	2·8441	52·647	0·8455
0·55	3·3752	50·238	0·9019
0·6	4·2147	48·299	0·9217
0·65	5·8293	46·589	0·9201
0·7	12·8630	45·186	0·9114
0·711	27·3166	44·907	0·8763
	θ_s = 45°		
0·3	2·3720	70·058	−0·4205
0·32	2·3910	68·077	−0·1833
0·3383	2·4336	66·381	0·0000
0·34	2·4387	66·228	0·0154
0·35	2·4718	65·354	0·1017
0·4	2·7233	61·487	0·4271
0·4082	2·7866	60·847	0·4709
0·45	3·1279	58·360	0·6220
0·5	3·7626	55·827	0·7311
0·55	4·8844	53·757	0·7937
0·6	7·7892	52·046	0·8261
0·633	26·2600	51·075	0·8640
	θ_s = 50°		
0·3	3·1392	71·261	−0·3833
0·32	3·1735	69·593	−0·1723
0·3395	3·2539	68·078	−0·0012
0·3396	3·2546	68·067	0·0000
0·35	3·3154	67·302	0·0780
0·4	3·7916	64·038	0·3675
0·4082	3·9220	63·497	0·4078
0·45	4·7239	61·385	0·5445
0·5	7·0019	59·225	0·6520
0·54	21·4992	57·782	0·8512
	θ_s = 55°		
0·3	5·4816	72·901	−0·3628
0·35	6·2273	69·588	0·0576
0·4	9·8838	66·850	0·3202
0·4082	12·0445	66·394	0·3573

TABLE 1 (*cont.*)